SIMS 255 Foundations of Software Design

Complexity and NP-completeness

Matt Welsh

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mdw@cs.berkeley.edu

Outline

Complexity of algorithms

- Space and time complexity
- "Big O" notation
- Complexity hierarchies and algorithm examples

$\mathcal{P}$ and $\mathcal{NP}$

- Decision problems
- Polynomial time decidability and computability
- The ultimate question: Does $\mathcal{P} = \mathcal{NP}$?
- $\mathcal{NP}$-completeness
- Examples of $\mathcal{NP}$-complete problems

Complexity of Algorithms

The kinds of questions we want to answer:

- Given an algorithm, how much time does it take to run?
- Given an algorithm, how much space does it use?
  - ▷ Characterized by the **size of the problem**

A simple example: finding max in a sequence of numbers

- Algorithm: Scan the numbers from 1 to N, find the maximum value
- The run time is "order N"

Another example: is a given number prime?

- Recall - prime number cannot be divided by any integer
- The "size of the problem" is $N = \log_{10} x$ (number of digits in x)
- Brute force algorithm: divide x by every number less than $\sqrt{x}$
- Run time: about $10^{N/2}$

Big O Notation

This is a way of characterizing the run time (or space constraints) of a given algorithm.

We say a function $f(n)$ is "$O(g(n))$" when

$$f(n) \leq Cg(n)$$

for some constant of C.

For example,

$$f(n) = 859398n^5 + 29810n^3 + 10191032n$$

is $O(n^5)$, since as $n \rightarrow \infty$, $f(n)$ "looks like" n^5 regardless of those big constants!

Big O Examples

Linear: $O(n)$

- e.g., Finding max or min in a sequence of numbers

Polynomial: $O(n^p)$ for some integer p

- Classic "bubblesort" algorithm is $O(n^2)$

Logarithmic: $O(\log n)$

- "Quicksort" algorithm is $O(n \log n)$
- To sort 1 million numbers, quicksort takes 6 million steps, bubblesort takes a trillion!

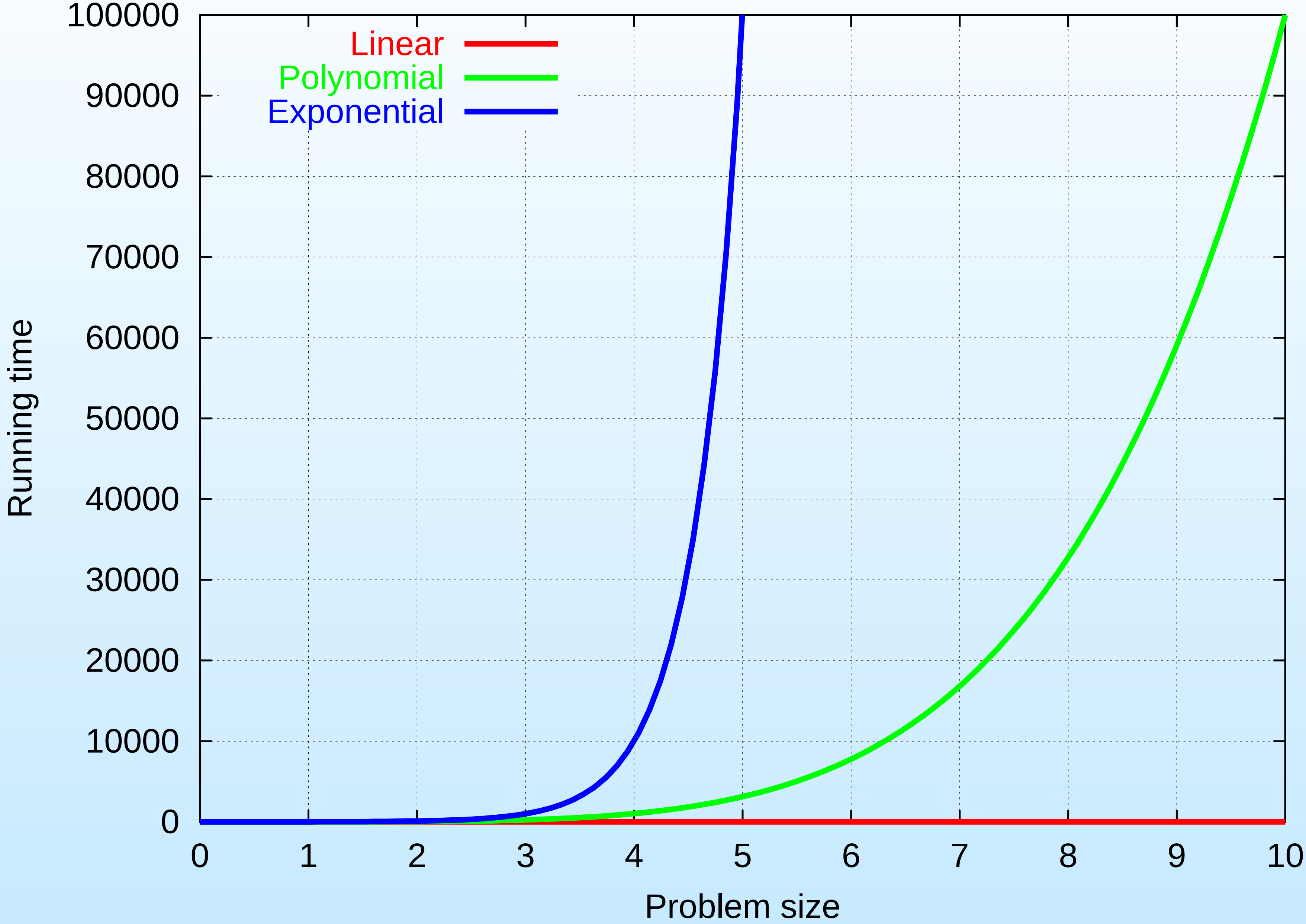
Exponential: $O(B^n)$ for some constant $B > 1$

- Brute force factorization, lots of numerical problems

Factorial: $O(n!)$

- $n!$ defined as $n \times (n-1) \times (n-2) \times ... \times 1$
- e.g., Calculating the Fibonacci numbers recursively (0, 1, 1, 2, 3, 5, 8, 13, ...)

Comparison of complexity classes



Tractable and Intractable Problems

Looking at the last slide, it seems that exponential run times are pretty bad!

- In fact, they are worse than almost anything else
- $O(n!)$ and $O(n^n)$ are even worse, but uncommon

We say that tractable problems are those that we can solve in practice

Intractible problems can be solved in theory, but not in practice

- Tractable problems have solutions that are polynomial or better
- Intractable problems have solutions that are exponential or worse

Some problems are flat-out unsolvable!

- This is not to say that they are "really hard", but rather no computer could possibly solve them
- Next lecture!

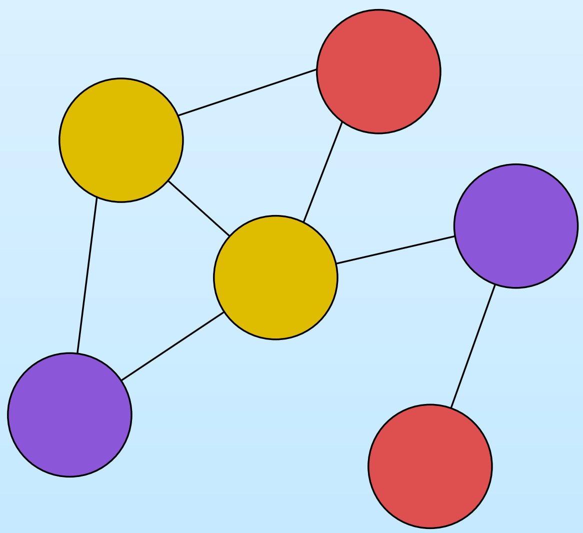
Complexity Classes

A **complexity class** is a set of computational problems with the same bounds in time and space

Say I give you a problem to solve -- what is your best hope for an efficient algorithm?

- If you can reduce the problem to another problem with known complexity, then you can answer straight away!

Example: Given a graph G, is it possible to color the nodes with just 3 colors, such that no two adjacent nodes have the same color?



Problem Reduction

It turns out we can reduce this problem to another one: **boolean satisfiability**

- Given a boolean expression of multiple variables, is there some assignment to the variables that makes the expression true?

$$(p \vee q \vee \overline{r} \vee s) \wedge (\overline{q} \vee s)$$

- What values should you assign to p, q, r, and s to make this statement true?

It turns out there is no known polynomial time solution to this problem!

Decision Problems

A decision problem is one that seeks a yes-or-no answer

Example: k-colorability

- Can this graph be colored with k colors?

Traveling Salesperson Problem

- Given a set of cities with distances between them, can someone travel to each city (without visiting a city more than once) in M miles or less?

traveling salesperson problem A classical scheduling problem that has baffled linear programmers for 30 years, but which, in a more complex formulation, is solved daily by traveling salespersons.

Optimization Problems

Some problems are optimization problems

- What is the **fewest number** of colors that color this graph?
- What is the **shortest path** that one can take to visit all the cities?

Usually if we have a way to solve the decision problem, without much more work we can solve the corresponding optimization problem:

- Start with a graph G
- Find out if G can be colored with 50 colors $\rightarrow$ yes or no
- If yes, then try 25 colors
- If yes, then try 12 colors, etc.

So, we mainly talk about decision problems

- Can easily derive the corresponding optimization problem

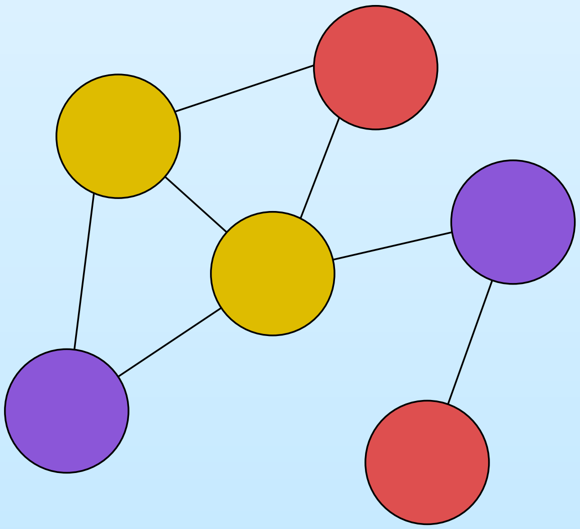
Polynomial-time decidability

We say a problem is polynomial-time decidable if:

- Given a problem P and a proposed solution s
- There is a polynomial time algorithm that checks whether s is a solution for P

Example: Graph colorability

- Given a graph G and an assignment of colors to nodes
- Can easily check whether any two nodes have the same color
- Simple algorithm is $O(n)$, where n is the number of nodes



The Complexity Class $\mathcal{NP}$

The set of problems for which the answer can be checked in polynomial time is called $\mathcal{NP}$

- $\mathcal{NP}$ stands for "nondeterministic polynomial time"

The name comes from a "nondeterministic Turing machine"

- Formal model of computing that allows the (theoretical) machine to perform an infinite number of operations simultaneously
- More about this next lecture!

For now, think of problems in $\mathcal{NP}$ as those that we have some way of quickly checking answers for

- But not necessarily a fast way to get the answer!

The Complexity Class $\mathcal{P}$

A problem is polynomial-time computable if:

- We can **find a solution** in polynomial time
- Note that this is quite different than **checking a given solution** !

The set of problems that have this property is called $\mathcal{P}$

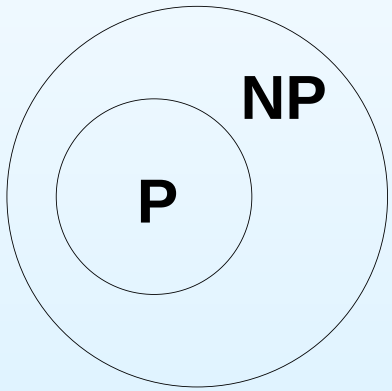
Generally speaking, problems in $\mathcal{P}$ are "good"

- i.e., We have fast algorithms for them
- Even if a problem is $O(n^{1902892})$, it's still better than $O(10^n)$!
- In practice, most problems in $\mathcal{P}$ are $O(n^k)$ for small k

Does $\mathcal{P} = \mathcal{NP}$?

All problems in $\mathcal{P}$ are also in $\mathcal{NP}$

- But the converse is not known to be true



The most important open problem in Computer Science !

- In fact there is a $1,000,000 award for anyone who can solve it

We know that many problems don't seem to have polynomial-time algorithms

- But, nobody has proven that these "hard" problems are **not in** $\mathcal{P}$
- There may be some mysterious poly-time algorithm for one of those "hard" problems lurking out there...

Example: Factoring Large Numbers

Many modern systems rely on public key cryptography

- Popular implementation is RSA
- Used in all Web browsers for secure connections

Start with two (large) prime numbers, and multiply them

- Primes P and Q, with product PQ
- Note that P and Q are the **only** two numbers that you can multiply to get PQ

We can make the product PQ public

- Because it is very hard to factor the number into the "secrets" P and Q!

Public key encryption idea:

- Bob publishes the number PQ to the world
- Any one can use PQ to encrypt a message for Bob
- Only Bob knows P and Q separately to decrypt the message

Factoring Large Numbers is Hard!

Factoring is in $\mathcal{NP}$

- It's easy to check whether two factors P and Q multiply to get PQ
- But, the fastest algorithm we have for finding factors is still exponential:

$$O(e^{c \log n^{1/3} \log \log n^{2/3}})$$

Still, better factoring algorithms are always being developed...

- In 1977, Ron Rivest said that factoring a 125-digit number would take 40 quadrillion years
- In 1994, a 129-digit number was factored

Upshot: If $\mathcal{P} = \mathcal{NP}$, then all hard problems (or at least those in $\mathcal{NP}$) can be solved in polynomial time!

- See the movie "Sneakers"

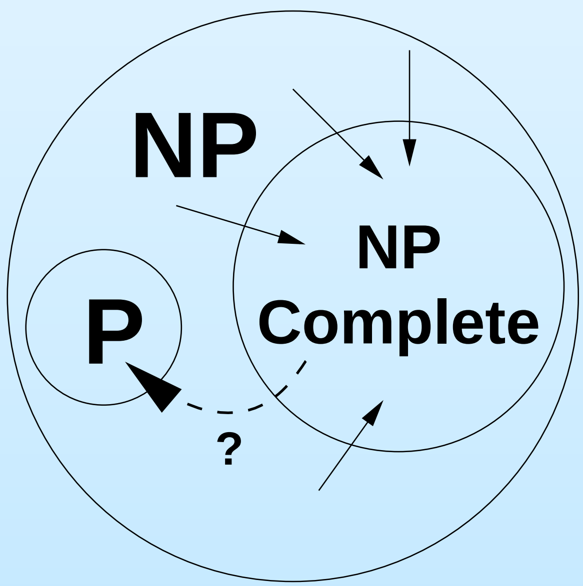
$\mathcal{NP}$-Completeness

The "hardest" problems in $\mathcal{NP}$ are called $\mathcal{NP}$-complete

A problem is $\mathcal{NP}$-complete if:

- It is in $\mathcal{NP}$
- All other problems in $\mathcal{NP}$ can be reduced to it (in polynomial time)

Result: If we can find a mapping from any $\mathcal{NP}$-complete problem to any problem in $\mathcal{P}$, then **all** problems in $\mathcal{NP}$ are also in $\mathcal{P}$!



Examples of $\mathcal{NP}$-Complete Problems

Boolean satisfiability

- Given a boolean expression in a set of variables, what values of the variables makes the expression true?

Traveling Salesperson Problem

- Given a set of cities connected by roads, what is the path of minimum distance that visits all cities exactly once?

k-colorability

- Given a graph G, what assignment of k colors to the nodes leaves no two adjacent nodes with the same color?

Partition problem

- Given a list of integers $x_1, x_2, ...$, does there exist a subset whose sum is exactly $\frac{1}{2}\sum x_i$?

The Deeper Meaning

$\mathcal{NP}$-completeness is about the theoretical limits of computing

- If a problem is $\mathcal{NP}$-complete, it is very unlikely that we will ever find a fast algorithm for it

Nobody knows whether $\mathcal{P} = \mathcal{NP}$

- Although many people have been working on it for years
- It's impressive that we can't even prove $\mathcal{P} \neq \mathcal{NP}$

This is not about Computer Scientists "not realizing" that there is a fast algorithm for an $\mathcal{NP}$-complete problem

- Rather, this is a fundamental limit on what can and cannot be computed efficiently!
- Huge implications: If you know a problem is $\mathcal{NP}$-complete, you might as well give up looking for a fast solution

Some hope for the future

Random algorithms and approximations

- Many $\mathcal{NP}$-complete problems can be approximated by fast techniques
- For example, Monte Carlo methods use randomness to "guess" an answer to a problem
- Can often trust the answer with 99.99999% (or more) confidence

Quantum Computing

- Computers built using quantum particles can quickly compute many answers simultaneously
- It turns out that quantum computers can (theoretically) solve many problems efficiently, for which no previous fast algorithm was known
- For example, a Quantum Computer can factor numbers in polynomial time!
- But, QCs are very hard to build

Summary

Algorithm complexity and "Big O" notation

Comparing complexities: linear, polynomial, exponential

Tractable and intractable problems

Complexity classes and decision problems

Polynomial-time decidability ($\mathcal{NP}$)

Polynomial-time computability ($\mathcal{P}$)

The $\mathcal{P} = \mathcal{NP}$ problem and $\mathcal{NP}$-completeness